

VARIATION OF WEYL MODULES IN p -ADIC FAMILIES

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ABSTRACT. Given a Weil-Deligne representation with coefficients in a domain, we prove the rigidity of the structures of the Frobenius-semisimplifications of the Weyl modules associated to its pure specializations. Moreover, we show that the structures of the Frobenius-semisimplifications of the Weyl modules attached to a collection of pure representations are rigid if these pure representations lift to Weil-Deligne representations over domains containing a domain \mathcal{O} and a pseudorepresentation over \mathcal{O} parametrizes the traces of these lifts.

1. INTRODUCTION

The aim of this article is to study the variation of the Weyl modules in families of automorphic Galois representations. We show that the variation of the Weyl modules is related to the purity of p -adic automorphic Galois representations at the places outside p and establish the rigidity of the structures of the Weyl modules at the arithmetic points of irreducible components of p -adic families.

Let p, ℓ be two distinct primes, K be a finite extension of \mathbb{Q}_ℓ and \mathcal{O} be an integral domain containing \mathbb{Q} . Let \mathbb{S}_μ denote the Schur functor corresponding to a partition μ of a positive integer d . In theorem 3.1, we show that the structures of the Frobenius-semisimplifications of the Weyl modules associated to (*i.e.*, \mathbb{S}_μ applied to) a collection of pure representations (*i.e.*, representations whose associated monodromy filtrations and weight filtrations coincide up to some shift) of the Weil group W_K over $\overline{\mathbb{Q}_p}$ are “rigid” if these pure representations are specializations of a Weil-Deligne representation over \mathcal{O} . More generally, we prove that the structures of the Frobenius-semisimplifications of the Weyl modules attached to a collection of pure representations of W_K over $\overline{\mathbb{Q}_p}$ are “rigid” if these representations lift to Weil-Deligne representations over domains containing \mathcal{O} and a pseudorepresentation $T : W_K \rightarrow \mathcal{O}$ parametrizes the traces of these lifts (see theorem 3.2).

The eigenvarieties provide examples of p -adic families of automorphic Galois representations. If T denotes the pseudorepresentation associated to an eigenvariety X , then for any nonempty admissible open affinoid subset U of X , the restriction of T to $\mathcal{O}(U)$ lifts to a Galois representation on a finite type module over some integral extension of the normalization of $\mathcal{O}(U)$ (by [1, Lemma 7.8.11]). However, this module is not known to be free. So theorem 3.2 cannot be applied to eigenvarieties to study the Weyl modules associated to all arithmetic points. To circumvent this problem, we prove theorem 3.3 which applies to the Weyl modules associated to the arithmetic points whose associated Galois representations are absolutely irreducible and give pure representations when restricted to decomposition groups at the places outside p . Theorem 3.1, 3.2, 3.3 are generalizations of [13, Theorem

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4.1, 5.4, 5.6] and hence apply to p -adic families of automorphic representations (in the same way as [13, Theorem 4.1, 5.4, 5.6] are applicable, see section 6 of *loc. cit.*) since automorphic Galois representations are known to be pure in many cases. We illustrate such an application in theorem 4.1, which explains the variation of the symmetric powers of the Galois representations attached to the arithmetic points of the Hida family of ordinary cusp forms.

2. PRELIMINARIES

For every field F , fix an algebraic closure \overline{F} of it and let G_F denote the absolute Galois group $\text{Gal}(\overline{F}/F)$. Given a domain A , let $Q(A)$ denote its fraction field, $\overline{Q}(A)$ denote the field $\overline{Q(A)}$, A^{intal} denote the integral closure of A in $\overline{Q}(A)$. For every map $f : A \rightarrow B$ between domains, fix an extension $f^{\text{intal}} : A^{\text{intal}} \rightarrow B^{\text{intal}}$ of f .

Let μ be a partition of a positive integer d . Let \mathfrak{S}_d denote the group of all permutations of the set $\{1, \dots, d\}$ and c_μ denote the Young symmetrizer in $\mathbb{Z}[\mathfrak{S}_d]$ attached to μ (see [8, §4.1] for more details). Then there exists a positive integer n_μ such that $c_\mu^2 = n_\mu c_\mu$ in $\mathbb{Z}[\mathfrak{S}_d]$ (see [8, Lemma 4.26]). For any module M over a commutative ring R , the R -module $M^{\otimes d}$ carries a left R -linear action of $\text{Aut}_R(M)$ and a right action of \mathfrak{S}_d given by $(m_1 \otimes \dots \otimes m_d) \cdot \sigma = m_{\sigma(1)} \otimes \dots \otimes m_{\sigma(d)}$. This right action commutes with the left action of $\text{Aut}_R(M)$. If n_μ is invertible in R , then denote the image of $M^{\otimes d}$ under c_μ by $\mathbb{S}_\mu M$, which is again an R -module and carries an R -linear left action of $\text{Aut}_R(M)$. We call the functor $M \mapsto \mathbb{S}_\mu M$ the *Schur functor* or *Weyl module* corresponding to μ . If V is a vector space over a field F of characteristic zero, then the dimension of $\mathbb{S}_\mu V$ is equal to $\dim_{\mathbb{Q}} \mathbb{S}_\mu(\mathbb{Q}^{\dim_F V})$. We denote this integer by $d(\mu, \dim_F V)$. Moreover, if M carries an R -linear action of a group G and n_μ is invertible in R , then $\mathbb{S}_\mu M$ also inherits an action of G , *i.e.*, given a representation $\rho : G \rightarrow \text{Aut}_R(M)$, the module $\mathbb{S}_\mu M$ can be considered as an R -linear representation of G via the composite map $G \xrightarrow{\rho} \text{Aut}_R(M) \rightarrow \text{Aut}_R(\mathbb{S}_\mu M)$. We denote by $\mathbb{S}_\mu \rho$ the R -module $\mathbb{S}_\mu M$ together with the R -linear action of G given by this composite map.

Remark 2.1. If μ is equal to the partition $d = d$ (resp. $d = 1 + \dots + 1$), then for any $\mathbb{Z}[1/n_\mu]$ -module M , the module $\mathbb{S}_\mu M$ is equal to $\text{Sym}^d M$ (resp. $\wedge^d M$). For example, if $\rho : G \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$ is a representation, then $\text{Sym}^d \rho$ denotes the $\overline{\mathbb{Q}}_p$ -vector space $\text{Sym}^d(\overline{\mathbb{Q}}_p^n)$ together with the $\overline{\mathbb{Q}}_p$ -linear action of G given by the composite map

$$G \xrightarrow{\rho} \text{GL}_n(\overline{\mathbb{Q}}_p) = \text{Aut}_{\overline{\mathbb{Q}}_p}(\overline{\mathbb{Q}}_p^n) \rightarrow \text{Aut}_{\overline{\mathbb{Q}}_p}(\text{Sym}^d(\overline{\mathbb{Q}}_p^n)).$$

Let k denote the residue field of the ring of integers \mathcal{O}_K of K and ϕ denote an element of G_K which maps to the geometric Frobenius element $\text{Fr}_k \in G_k$. The Weil group W_K is defined as the subgroup of G_K consisting of elements which map to integral powers of Fr_k in G_k . Define $v_K : W_K \rightarrow \mathbb{Z}$ by $\sigma|_{K^{\text{ur}}} = \text{Fr}_k^{v_K(\sigma)}$ for all $\sigma \in W_K$. In the following, A denotes a commutative integral domain of characteristic zero.

Definition 2.2 ([4, 8.4.1]). A Weil-Deligne representation of W_K on a free module M of finite type over A is a pair (r, N) consisting of a representation $r : W_K \rightarrow \text{Aut}_A(M)$ with open kernel and a nilpotent element N of $\text{End}_A(M)$ such that

$$r(\sigma)Nr(\sigma)^{-1} = (\#k)^{-v_K(\sigma)}N$$

for any $\sigma \in W_K$. A representation ρ of W_K on M is said to be irreducible Frobenius-semisimple if ρ has open kernel, $M \otimes \overline{\mathbb{Q}}(A)$ is irreducible and the ϕ -action on $M \otimes \overline{\mathbb{Q}}(A)$ is semisimple.

Given a Weil-Deligne representation (r, N) of W_K on a vector space V with coefficients in an algebraically closed field of characteristic zero, we denote its Frobenius-semisimplification by $(r, N)^{\text{Fr-ss}}$ (see [4, p. 570]).

Definition 2.3. Given Weil-Deligne representations (r_1, N_1) on an A -module M_1 and (r_2, N_2) on an A -module M_2 , their tensor product is defined as the Weil-Deligne representation $(r_1 \otimes r_2, \text{id}_{M_1} \otimes N_2 + N_1 \otimes \text{id}_{M_2})$ on $M_1 \otimes_A M_2$.

Definition 2.4. Let (r, N) be a Weil-Deligne representation of W_K on a free module M over a domain R containing \mathbb{Q} . If $\mathbb{S}_\mu M$ is free over R , then the Weil-Deligne representation $\mathbb{S}_\mu(r, N)$ is defined as $(r, N)^{\otimes d}|_{\mathbb{S}_\mu M}$.

3. CONTROL THEOREMS FOR WEYL MODULES

The following results are the analogues of [13, Theorem 4.1, 5.4, 5.6] in the context of Weyl modules.

Theorem 3.1. Let $(r, N) : W_K \rightarrow \text{GL}_n(\mathcal{O})$ be a Weil-Deligne representation. Let m, t_1, \dots, t_m be positive integers, r_1, \dots, r_m be irreducible Frobenius-semisimple representations of W_K over $\mathcal{O}^{\text{intal}}$ such that

$$(3.1) \quad (\mathbb{S}_\mu((r, N) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}(\mathcal{O})))^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^m \text{Sp}_{t_i}(r_i).$$

If $f \circ (r, N)$ is pure for some map $f : \mathcal{O} \rightarrow \overline{\mathbb{Q}}_p$, then the Weil-Deligne representations $(\mathbb{S}_\mu(f \circ (r, N)))^{\text{Fr-ss}}$ and $\bigoplus_{i=1}^m \text{Sp}_{t_i}(f^\dagger \circ r_i)$ are isomorphic for any lift $f^\dagger : \mathcal{O}^{\text{intal}} \rightarrow \overline{\mathbb{Q}}_p$ of f . Moreover, there exist m, t_i, r_i with the above-mentioned properties such that equation (3.1) holds.

Proof. Let \mathcal{T} denote the \mathcal{O} -module \mathcal{O}^n and \mathfrak{p} denote the kernel of f . Note that $\mathcal{T}_{\mathfrak{p}}^{\otimes d}$ decomposes as the direct sum $\mathbb{S}_\mu(\mathcal{T}_{\mathfrak{p}}) \oplus \mathcal{T}_{\mathfrak{p}}^{\otimes d}(n_\mu - c_\mu)$ (as $\text{End}_{\mathcal{O}}(\mathcal{T})$ -modules). Moreover these summands are free over $\mathcal{O}_{\mathfrak{p}}$ by [10, Proposition 3.G]. So $\mathbb{S}_\mu(\mathcal{T}_{\mathfrak{p}}) \otimes_{\mathcal{O}_{\mathfrak{p}}, f} \overline{\mathbb{Q}}_p$ is isomorphic to its image in $(\mathcal{T}^{\otimes d})_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}, f} \overline{\mathbb{Q}}_p$, i.e., to $\mathbb{S}_\mu(f \circ (r, N))$ as Weil-Deligne representations. Since $(f \circ (r, N))^{\otimes d}$ is pure (by [6, Proposition 1.6.9]), the representation $\mathbb{S}_\mu(f \circ (r, N))$ is also pure. So the result follows from [13, Theorem 4.1]. \square

Theorem 3.2. Let $T : W_K \rightarrow \mathcal{O}$ be a pseudorepresentation. Let $(r, N) : W_K \rightarrow \text{GL}_n(\mathcal{O})$ be a Weil-Deligne representation over a domain \mathcal{O} such that $\text{res} \circ T = \text{trr}$ for an injective map $\text{res} : \mathcal{O} \hookrightarrow \mathcal{O}$. Suppose m, t_1, \dots, t_m are positive integers, r_1, \dots, r_m are irreducible Frobenius-semisimple representations of W_K over $\mathcal{O}^{\text{intal}}$ such that

$$(3.2) \quad (\mathbb{S}_\mu((r, N) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}(\mathcal{O})))^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^m \text{Sp}_{t_i}(\text{res}^{\text{intal}} \circ r_i).$$

Suppose $f \circ (r, N)$ is pure for some map $f : \mathcal{O} \rightarrow \overline{\mathbb{Q}}_p$. Then for any Weil-Deligne representation $(r', N') : W_K \rightarrow \text{GL}_n(\mathcal{O}')$ over a domain \mathcal{O}' such that trr' is equal to $\text{res}' \circ T$ for some

injective map $\text{res}' : \mathcal{O} \hookrightarrow \mathcal{O}'$ and $f' \circ (r', N')$ is pure for some map $f' : \mathcal{O}' \rightarrow \overline{\mathbb{Q}}_p$, the Weil-Deligne representations $(\mathbb{S}_\mu((r', N') \otimes_{\mathcal{O}'} \overline{\mathbb{Q}}(\mathcal{O}')))^{\text{Fr-ss}}, (\mathbb{S}_\mu(f' \circ (r', N')))^{\text{Fr-ss}}$ are isomorphic to $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\text{res}'^\dagger \circ r_i), \bigoplus_{i=1}^m \text{Sp}_{t_i}(f'^\dagger \circ \text{res}'^\dagger \circ r_i)$ respectively for any lift $\text{res}'^\dagger : \mathcal{O}^{\text{intal}} \rightarrow \mathcal{O}'^{\text{intal}}$ of res' and $f'^\dagger : \mathcal{O}'^{\text{intal}} \rightarrow \overline{\mathbb{Q}}_p$ of f' . Furthermore, there exist m, t_i, r_i with the above-mentioned properties such that equation (3.2) holds.

Proof. Let ρ be a representation of W_K over $\overline{\mathbb{Q}}(\mathcal{O})$ such that its trace is equal to T . Let $\mathfrak{p}, \mathfrak{p}'$ denote the kernel of f, f' respectively. Note that $\mathbb{S}_\mu((r, N) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}})$ is a Weil-Deligne representation on $\mathcal{O}_{\mathfrak{p}}^{d(\mu, n)}$ and $\mathbb{S}_\mu(f \circ (r, N)) = f \circ (\mathbb{S}_\mu((r, N) \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}))$ is pure by [6, Proposition 1.6.9]. Since $\text{res}^{\text{intal}} \circ \rho$ and r have equal traces, the representations $\text{res}^{\text{intal}} \circ \mathbb{S}_\mu \rho, \mathbb{S}_\mu r$ also have equal traces (by [8, Theorem 6.3.(3)], for instance) and hence $\text{res}^{\text{intal}} \circ \text{tr} \mathbb{S}_\mu \rho$ is equal to the trace of $\mathbb{S}_\mu r$. Similarly, $\mathbb{S}_\mu((r', N') \otimes_{\mathcal{O}'} \mathcal{O}'_{\mathfrak{p}'})$ is a Weil-Deligne representation on $\mathcal{O}'_{\mathfrak{p}'}^{d(\mu, n)}$, the representation $\mathbb{S}_\mu(f' \circ (r', N')) = f' \circ (\mathbb{S}_\mu((r', N') \otimes_{\mathcal{O}'} \mathcal{O}'_{\mathfrak{p}'}))$ is pure and $\text{res}'^{\text{intal}} \circ \text{tr} \mathbb{S}_\mu \rho$ is equal to the trace of $\mathbb{S}_\mu r'$. Also note that $\text{tr} \mathbb{S}_\mu \rho$ is an $\mathcal{O}^{\text{intal}}$ -valued pseudorepresentation of W_K . Thus the result follows from [13, Theorem 5.4]. \square

Suppose \mathcal{O} is a \mathbb{Z}_p -algebra and let $w \nmid p$ denote a finite place of a number field F . Let T, T_1, \dots, T_n be \mathcal{O} -valued pseudorepresentations of G_F such that $T = T_1 + \dots + T_n$. Using [16, Theorem 1], we choose semisimple representations $\sigma_1, \dots, \sigma_n$ of G_F over $\overline{\mathbb{Q}}(\mathcal{O})$ such that $\text{tr} \sigma_i = T_i$ for all $1 \leq i \leq n$. The *irreducibility and purity locus* of T_1, \dots, T_n is defined to be the collection of tuples $(\mathcal{O}, \mathfrak{m}, \kappa, \text{loc}, \rho_1, \dots, \rho_n)$ where \mathcal{O} is a Henselian Hausdorff domain and it is a \mathbb{Z}_p -algebra, \mathfrak{m} denotes its maximal ideal, κ denotes its residue field which is an algebraic extension of \mathbb{Q}_p , $\text{loc} : \mathcal{O} \hookrightarrow \mathcal{O}$ is an injective \mathbb{Z}_p -algebra map, ρ_1, \dots, ρ_n are irreducible G_F -representations over $\overline{\kappa}$ such that their traces are equal to $\text{loc} \circ T_1 \bmod \mathfrak{m}, \dots, \text{loc} \circ T_n \bmod \mathfrak{m}$ respectively and their restrictions to the decomposition group of F at w are pure. Given such a tuple $(\mathcal{O}, \mathfrak{m}, \kappa, \text{loc}, \rho_1, \dots, \rho_n)$, we use [11, Théorème 1] to choose semisimple G_F -representations $\tilde{\rho}_1, \dots, \tilde{\rho}_n$ over \mathcal{O} such that $\text{tr} \tilde{\rho}_i = \text{loc} \circ T_i$ for all $1 \leq i \leq n$.

Theorem 3.3. *Assume that the restrictions of $\sigma_1, \dots, \sigma_n$ to the Weil group W_w of F at w are potentially unipotent. Then there exist positive integers m, t_1, \dots, t_m , irreducible Frobenius-semisimple representations r_1, \dots, r_m of W_w over $\mathcal{O}^{\text{intal}}$ such that*

$$(3.3) \quad \text{WD} \left(\bigoplus_{i=1}^n \mathbb{S}_\mu \sigma_i|_{W_w} \right)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^m \text{Sp}_{t_i}(r_i)_{/\overline{\mathbb{Q}}(\mathcal{O})}$$

and there are isomorphisms of Weil-Deligne representations

$$(3.4) \quad \text{WD} \left(\bigoplus_{i=1}^n \mathbb{S}_\mu \rho_i|_{W_w} \right)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^m \text{Sp}_{t_i}(\pi_{\mathfrak{m}}^{\text{intal}} \circ \text{loc}^{\text{intal}} \circ r_i)_{/\overline{\kappa}},$$

$$(3.5) \quad \text{WD} \left(\bigoplus_{i=1}^n \mathbb{S}_\mu \tilde{\rho}_i|_{W_w} \otimes \overline{\mathbb{Q}}(\mathcal{O}) \right)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^m \text{Sp}_{t_i}(\text{loc}^{\text{intal}} \circ r_i)_{/\overline{\mathbb{Q}}(\mathcal{O})}$$

for any element $(\mathcal{O}, \mathfrak{m}, \kappa, \text{loc}, \rho_1, \dots, \rho_n)$ in the irreducibility and purity locus of T_1, \dots, T_n where $\pi_{\mathfrak{m}}$ denotes the mod \mathfrak{m} reduction map $\mathcal{O} \rightarrow \mathcal{O}/\mathfrak{m}$.

Proof. If the irreducibility and purity locus of T_1, \dots, T_n is empty, then it remains to prove equation (3.3), which follows from [3, Proposition 3.1.3 (i)]. So we assume that this locus

is nonempty. Note that the representation $\tilde{\rho}_i$ is irreducible since the mod \mathfrak{m} reduction of its trace is equal to the trace of the irreducible representation ρ_i . Since the G_F -representations $\sigma_i \otimes \overline{\mathbb{Q}}(\mathcal{O}), \tilde{\rho}_i$ have equal traces, they are isomorphic. Since $\sigma_i|_{W_w}$ is potentially unipotent, the representation $\tilde{\rho}_i|_{W_w}$ is also potentially unipotent. Consequently, its Weil-Deligne parametrization $\text{WD}(\tilde{\rho}_i|_{W_w})$ is defined and has coefficients in \mathcal{O} . Moreover, the trace of $\text{WD}(\tilde{\rho}_i|_{W_w})$ is equal to $\text{loc} \circ T_i|_{W_w} = \text{loc} \circ \text{tr} \sigma_i$ and the mod \mathfrak{m} reduction of $\text{WD}(\tilde{\rho}_i|_{W_w})$ is isomorphic to the pure representation $\text{WD}(\rho_i|_{W_w})$. So the isomorphisms in equation (3.4) and (3.5) follow from theorem 3.2. Then equation (3.5) gives equation (3.3) since $\sigma_i \otimes \overline{\mathbb{Q}}(\mathcal{O})$ is isomorphic to $\tilde{\rho}_i$. \square

4. WEYL MODULES IN FAMILIES

In this section, we prove a control theorem for the symmetric powers of the Galois representations attached to arithmetic points of Hida family ordinary cusp forms. To prove this result, we use purity of Galois representations attached to cusp forms.

Given a normalized eigen cusp form $f = \sum_{n=1}^{\infty} a_n q^n$ of weight at least two, from the works of Eichler [7], Shimura [15], Deligne [5] and Ribet [12, Theorem 2.3], it follows that there exists a continuous Galois representation $\rho_f : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$ (unique up to equivalence) such that the trace of $\rho_f(\text{Fr}_{\ell})$ is equal to a_{ℓ} for any prime ℓ not dividing the product of p and the level of f .

Let p be an odd prime, N be a positive integer such that $Np \geq 4$ and $p \nmid N$. Choose a minimal prime ideal \mathfrak{a} of the universal p -ordinary Hecke algebra h^{ord} of tame level N (denoted $h^{\text{ord}}(N; \mathbb{Z}_p)$ in [9]). The ring h^{ord} is an algebra over $\mathbb{Z}_p[[X]]$. An *arithmetic specialization* of h^{ord} is a \mathbb{Z}_p -algebra map $\lambda : h^{\text{ord}} \rightarrow \overline{\mathbb{Q}}_p$ such that $\lambda((1+X)^{p^r} - (1+p)^{(k-2)p^r}) = 0$ for some integers $k \geq 2$ and $r \geq 0$. Denote the quotient ring $h^{\text{ord}}/\mathfrak{a}$ by $R(\mathfrak{a})$, its fraction field by $\mathcal{Q}(\mathfrak{a})$ and fix an algebraic closure $\overline{\mathcal{Q}}(\mathfrak{a})$ of $\mathcal{Q}(\mathfrak{a})$. By [9, Theorem 3.1], there exists a unique (up to equivalence) continuous (in the sense of [9, §3]) absolutely irreducible two-dimensional Galois representation $\rho_{\mathfrak{a}}$ of $G_{\mathbb{Q}}$ over $\overline{\mathcal{Q}}(\mathfrak{a})$ with traces in $R(\mathfrak{a})$ and satisfying $\text{tr}(\rho_{\mathfrak{a}}(\text{Fr}_{\ell})) = T_{\ell} \bmod \mathfrak{a}$ for all prime ℓ not dividing Np where $T_{\ell} \in h^{\text{ord}}$ denotes the Hecke operator associated to the prime ℓ . By the isomorphism of [9, Theorem 2.2], there is a one-to-one correspondence between the arithmetic specializations of h^{ord} and the p -ordinary p -stabilized (in the sense of [17, p. 538]) normalized eigen cusp forms of weight at least 2 and tame level a divisor of N . Moreover, the trace of $\rho_{f_{\lambda}}$ is equal to $\lambda \circ \text{tr} \rho_{\mathfrak{a}}$ for any arithmetic specialization λ of h^{ord} with $\lambda(\mathfrak{a}) = 0$ where f_{λ} denotes the ordinary form associated to λ .

Theorem 4.1. *Let $\ell \neq p$ be a prime and W_{ℓ} denote the Weil group of \mathbb{Q}_{ℓ} . Then there exist positive integers m, t_1, \dots, t_m and irreducible Frobenius-semisimple representations r_1, \dots, r_m of W_{ℓ} with coefficients in $R(\mathfrak{a})^{\text{intal}}[1/p]$ such that $\text{WD}(\text{Sym}^d \rho_{\mathfrak{a}}|_{W_{\ell}})^{\text{Fr-ss}}$ is isomorphic to $\bigoplus_{i=1}^m \text{Sp}_{t_i}(r_i)$ and for any arithmetic specialization λ of $R(\mathfrak{a})$, the representations $\text{WD}(\text{Sym}^d \rho_{\lambda}|_{W_{\ell}})^{\text{Fr-ss}}$ and $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\lambda^{\text{intal}} \circ r_i)$ are isomorphic.*

Proof. Since the representation $\rho_{\mathfrak{a}}$ is continuous and $\ell \neq p$, by Grothendieck's monodromy theorem (see [14, p. 515–516]), the action of the inertia subgroup I_{ℓ} on $\rho_{\mathfrak{a}}$ is potentially unipotent. So the Weil-Deligne parametrization $\text{WD}(\rho_{\mathfrak{a}}|_{W_{\ell}})$ of $\rho_{\mathfrak{a}}|_{W_{\ell}}$ is defined and has coefficients in $R(\mathfrak{a})[1/p]$. Its λ -specialization is isomorphic to $\text{WD}(\rho_{f_{\lambda}})$, which is pure by [2]. Then the result follows from theorem 3.1. \square

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